

CUP-PRODUCTS IN GENERALIZED MOMENT-ANGLE COMPLEXES

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ABSTRACT. Given a family of based CW-pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ together with an abstract simplicial complex K with m vertices, there is an associated based CW-complex $Z(K; (\underline{X}, \underline{A}))$ known as a generalized moment-angle complex [1].

The decomposition theorem of [1], [2] splits the suspension of $Z(K; (\underline{X}, \underline{A}))$ into a bouquet of spaces determined by the full sub-complexes of K . That decomposition theorem is used here to describe the ring structure for the cohomology of $Z(K; (\underline{X}, \underline{A}))$. Explicit computations are made for families of suspension pairs and for the cases where X_i is the cone on A_i . These results complement and generalize those of Davis-Januszkiewicz [5], Franz, [7] and [6], Hochster [8] as well as Panov [9] and Baskakov-Buchstaber-Panov, [3]. Under conditions stated below, these theorems also apply for generalized cohomology theories.

1. Introduction, definitions, and main results

This paper is a study of the cup-product structure for the cohomology ring of a generalized moment-angle complex. The new result here is that the structure of the cohomology ring is given in terms of a geometric decomposition arising after one suspension of the generalized moment-angle complex [1, 2].

This cup-product structure was studied for special cases in [5, 3, 4, 8, 9, 7, 6] with important basic cases first given by Davis-Januszkiewicz [5], Franz [7, 6], and Buchstaber-Panov [4, 9]. A few details concerning historical developments are listed in [1, 2]. The methods here give a determination of the cohomology ring structure for many new generalized moment-angle complexes as well as retrieve many known results.

A generalized moment-angle complex is a union of cartesian products of based CW-complexes [1, 2]. Generalized moment-angle complexes satisfy a stable decomposition extending a classical decomposition for suspensions of product spaces. This decomposition informs on the cup product structure by providing information about the diagonal map,

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after stabilization; this suffices to give the cup-product structure. The notation used in this article is adopted from [1, 2].

Consider a product of based CW-complexes $Y^{[m]} = \prod_{i=1}^m Y_i$. Let $I \subseteq [m]$ be an increasing subsequence of $[m] = (1, 2, \dots, m)$. If $I = (i_1, \dots, i_k)$, then \widehat{Y}^I denotes the smash product $Y_{i_1} \wedge \dots \wedge Y_{i_k}$ the quotient space of $Y^I = Y_{i_1} \times \dots \times Y_{i_k}$ by the subspace given by the fat wedge $FW(Y^I) = \{(y_{i_1}, \dots, y_{i_k}) \in Y^I \mid y_{i_j} = \text{base-point of } Y_{i_j} \text{ for at least one } i_j\}$.

Recall that a generalized moment-angle complex is a functor of two variables:

- (1) An abstract simplicial complex K with m vertices identified with the sequence $(1, \dots, m)$. Then the simplices of K are identified with increasing subsequences $\sigma = (i_1, \dots, i_t)$. The dimension of σ is $t - 1$. The defining property of K is that if $\tau \subset \sigma$ is a subsequence of σ then $\tau \in K$. The empty set \emptyset belongs to K .
- (2) A family $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i \in [m]}$ of connected, based CW-pairs (X_i, A_i, x_i) .

The morphisms in (1) are embeddings of simplicial complexes and the morphisms in (2) are maps of based connected pairs $(\underline{X}, \underline{A}, \underline{x}) \rightarrow (\underline{Y}, \underline{B}, \underline{y})$. The simplicial complex K has a family of full sub-complexes K_I defined for every subsequence I of $[m]$,

$$K_I = \{\sigma \cap I \mid \sigma \in K\}$$

K_I has $l(I)$, length of I , vertices,

and associated to K_I a family of spaces $(\underline{X}, \underline{A})_I = \{(X_i, A_i)\}_{i \in I}$. If $I = [m]$, then $K_I = K$.

Next, define the functors $Z(K; (\underline{X}, \underline{A}))$ and $\widehat{Z}(K; (\underline{X}, \underline{A}))$ as in [1, 2] in the following way. For every $\sigma \in K$, define

$$D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$$

with $D(\emptyset) = A_1 \times \dots \times A_m$. As in [1], the space $Z(K; (\underline{X}, \underline{A}))$ is defined as $Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma)$.

In what follows, it is useful to define variations for a fixed, ambient I where the analogue of $D(\sigma)$ is replaced as follows.

Definition 1.1. For fixed $I = (i_1, \dots, i_k)$, and every $\sigma \in K$, define

$$(1.1) \quad Y^I(\sigma \cap I) = Y_{i_1} \times \dots \times Y_{i_k},$$

and

$$(1.2) \quad \widehat{Y}^I(\sigma \cap I) = Y_{i_1} \wedge \dots \wedge Y_{i_k}$$

where

$$(1.3) \quad Y_j = \begin{cases} X_j & \text{if } j \in \sigma \cap I \\ A_j & \text{if } j \in I - \sigma \cap I. \end{cases}$$

Furthermore,

$$\begin{aligned} Y^I(\Phi) &= A^I = A_{i_1} \times \cdots \times A_{i_k} \\ \widehat{Y}^I(\Phi) &= \widehat{A}^I = A_{i_1} \wedge \cdots \wedge A_{i_k}. \end{aligned}$$

Then the generalized moment-angle complexes are

$$Z(K_I; (\underline{X}, \underline{A})_I) = \bigcup_{\sigma \in K} Y^I(\sigma \cap I)$$

and

$$\widehat{Z}(K_I; (\underline{X}, \underline{A})_I) = \bigcup_{\sigma \in K} \widehat{Y}^I(\sigma \cap I).$$

Note: The notation $Z(K_I; (\underline{X}_I, \underline{A}_I))$ was used in [1] for $Z(K_I; (\underline{X}, \underline{A})_I)$.

To simplify notation below, the following notation

$$Z(K), \widehat{Z}(K), Z(K_I) \text{ and } \widehat{Z}(K_I)$$

is used to denote $Z(K; (\underline{X}, \underline{A}))$, $\widehat{Z}(K; (\underline{X}, \underline{A}))$, $Z(K_I; (\underline{X}, \underline{A})_I)$, and $\widehat{Z}(K_I; (\underline{X}, \underline{A})_I)$ respectively.

The results of [1] stated next are the main ingredients used here to analyze the cup-product structure for the generalized moment-angle complex.

Theorem 1.2. *Let K be an abstract simplicial complex with m vertices. Assume that*

$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$$

are pointed triples of CW-complexes for all i . Then there is a natural, pointed homotopy equivalence

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X}, \underline{A})_I)\right).$$

Cup-products in the cohomology of any space W are induced by the diagonal map

$$W \rightarrow W \times W.$$

The main direction of this paper is an analysis of the behavior of the diagonal map for the generalized moment-angle complex and the properties of the diagonal map which are preserved by the stable decomposition of Theorem 1.2 above [1].

Let

$$\Delta_I : Y^I \rightarrow Y^I \wedge Y^I$$

denote the reduced diagonal of Y^I and let

$$\widehat{\Delta}_I : \widehat{Y}^I \rightarrow \widehat{Y}^I \wedge \widehat{Y}^I$$

denote the reduced diagonal of \widehat{Y}^I . In this paper *partial diagonals* are defined below

$$\widehat{\Delta}_I^{J,L} : \widehat{Y}^I \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L,$$

and by restriction

$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \rightarrow \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$

where $J \cup L = I$. If $I = J = L$, these maps coincide with the *reduced* diagonal $\widehat{\Delta}_I$. Furthermore, if $\widehat{\Pi}_I : Y^{[m]} \rightarrow \widehat{Y}^I$ is the projection, there are commutative diagrams of CW-complexes and based continuous maps

$$(1.4) \quad \begin{array}{ccc} Y^{[m]} & \xrightarrow{\Delta_{[m]}} & Y^{[m]} \wedge Y^{[m]} \\ \widehat{\Pi}_I \downarrow & & \downarrow \widehat{\Pi}_J \wedge \widehat{\Pi}_L \\ \widehat{Y}^I & \xrightarrow{\widehat{\Delta}_I^{J,L}} & \widehat{Y}^J \wedge \widehat{Y}^L \end{array}$$

and by restriction to $Z(K) \subset X^{[m]}$

$$(1.5) \quad \begin{array}{ccc} Z(K) & \xrightarrow{\Delta_K} & Z(K) \wedge Z(K) \\ \widehat{\Pi}_I \downarrow & & \downarrow \widehat{\Pi}_J \wedge \widehat{\Pi}_L \\ \widehat{Z}(K_I) & \xrightarrow{\widehat{\Delta}_I^{J,L}} & \widehat{Z}(K_J) \wedge \widehat{Z}(K_L). \end{array}$$

A definition is given next.

Definition 1.3. Assume given a family of based CW-pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$. Given cohomology classes $u \in H^p(Z(K_J)), v \in H^q(Z(K_L))$, define

$$u * v = (\widehat{\Delta}_I^{J,L})^*(u \otimes v) \quad \text{thus} \quad u * v \in H^{p+q}(\widehat{Z}(K_I)).$$

The element $u * v \in H^{p+q}(\widehat{Z}(K_I))$ is called the $*$ -product. Commutativity of diagram (1.5) gives

$$(1.6) \quad \widehat{\Pi}_I^*(u * v) = \widehat{\Pi}_J^*(u) \smile \widehat{\Pi}_L^*(v)$$

where \smile denotes the cup-product for the CW-complex $Z(K)$.

Let

$$\mathcal{H}^q(K; (\underline{X}, \underline{A})) = \bigoplus_{I \subseteq m} H^q(\widehat{Z}(K_I))$$

with

$$\mathcal{H}^*(K; (\underline{X}, \underline{A})) = \bigoplus_{I \subseteq m} H^*(\widehat{Z}(K_I)).$$

Define a map

$$\eta : \mathcal{H}^*(K; (\underline{X}, \underline{A})) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})))$$

where η restricted to $H^*(\widehat{Z}(K_I))$ is $\widehat{\Pi}_I^*$.

By the decomposition given in Theorem 2.8 of [1], $\eta = \bigoplus_{I \subseteq [m]} \Pi_I^*$ is an additive isomorphism.

The $*$ -product gives $\mathcal{H}^*(K; (\underline{X}, \underline{A}))$ the structure of an algebra, a fact which is checked in Section 3 where the next result is proven.

Theorem 1.4. *Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs. Then*

$$\eta : \mathcal{H}^*(K; (\underline{X}, \underline{A})) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})))$$

is a ring isomorphism.

Definition 1.5. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs. The pair $(\underline{X}, \underline{A})$ is a *suspension pair* if $(X_i, A_i) = (\Sigma(U_i), \Sigma(V_i))$ for each i with each inclusion $A_i \subset X_i$ given as the suspension of a map $f_i : V_i \rightarrow U_i$.

If the pair $(\underline{X}, \underline{A})$ is a suspension pair, then the reduced diagonal

$$\Delta_i : Y_i \rightarrow Y_i \wedge Y_i$$

for $Y_i = X_i$ or A_i is null-homotopic. This fact will be used below to prove the next result.

Theorem 1.6. *Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs. If $(\underline{X}, \underline{A}) = \{(\Sigma(U_i), \Sigma(V_i))\}_{i \in [m]}$ is a suspension pair, and $J \cap L \neq \emptyset$, then*

$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \rightarrow \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$

*is null-homotopic, and thus $u * v = 0$ for classes $u \in H^p \widehat{Z}(K_J)$ and $v \in H^q \widehat{Z}(K_L)$.*

Definition 1.7. Define two CW-complexes X and Y to be *stably wedge equivalent*, if (i) X is stably equivalent to $X_1 \vee \cdots \vee X_k$, (ii) Y is stably equivalent to $Y_1 \vee \cdots \vee Y_k$, and (iii) X_i is stably equivalent to Y_i for $i = 1, \dots, k$. In particular, if X and Y are stably wedge equivalent, then they are stably homotopy equivalent. Let $T = (t_1, \dots, t_m)$ be a sequence of positive integers, then define

$$(\underline{\Sigma^T X}, \underline{\Sigma^T A}) = \{(\Sigma^{t_i} X_i, \Sigma^{t_i} A_i)\}_{i=1}^m$$

where $\Sigma^{t_i} X_i$ is the t_i -th iterated suspension of X_i . Let $T = (t_1, \dots, t_m)$, and $T' = (t'_1, \dots, t'_m)$ denote two sequences of strictly positive integers of length m , and define $T \equiv T' \pmod 2$ if $t_i \equiv t'_i \pmod 2$ for all i .

Theorem 1.8. *Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs and that $T = (t_1, \dots, t_m)$, and $T' = (t'_1, \dots, t'_m)$ are two sequences of strictly positive integers of length m . Then the following hold.*

- (1) $Z(K; (\underline{\Sigma^T X}, \underline{\Sigma^T A}))$ and $Z(K; (\underline{\Sigma^{T'} X}, \underline{\Sigma^{T'} A}))$ are stably wedge equivalent.
- (2) If $T \equiv T' \pmod 2$, then $Z(K; (\underline{\Sigma^T X}, \underline{\Sigma^T A}))$ and $Z(K; (\underline{\Sigma^{T'} X}, \underline{\Sigma^{T'} A}))$ have isomorphic cohomology rings regarded as ungraded rings.

Theorem 1.9. *Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{CX}, \underline{X}) = \{(CX_i, X_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs such that the finite product*

$$(X_1 \times \cdots \times X_m) \times (Z(K_{I_1}; (D^1, S^0)) \times \cdots \times Z(K_{I_t}; (D^1, S^0)))$$

for all $I_j \subseteq [m]$ satisfies the strong form of the Künneth theorem. Then the cup-product structure for the cohomology algebra $H^(Z(K; (\underline{CX}, \underline{X})))$ is a functor of the cohomology algebras of X_i for all i , and $Z(K_I; (D^1, S^0))$ for all I .*

The analogue of Theorem 1.9 in the case of $H^*(Z(K; (\underline{X}, \underline{A})))$ for which A is contractible is given as Theorem 2.35 of [1]. In this case, the result depends only on the structure of the cohomology algebra of $H^*(X_i)$, $1 \leq i \leq m$.

The cohomology ring of $Z(K; (D^2, S^1))$ was studied by Panov [9] as well as others [3, 4, 6]. Panov proved that the cohomology ring was the Tor-algebra of the face ring of K as studied by Hochster [8]. Theorem 1.9 gives information about that ring.

Remark 1.10. Theorems 1.4, 1.6, and 1.8 above apply more generally to any cohomology theory. On the other hand, Theorem 1.9 applies in the case of any cohomology theory which satisfies additional restrictions formalized as follows.

Definition 1.11. A family of based CW-pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ together with a finite simplicial complex K with m vertices is said to be **proper with respect to a cohomology theory** $E^*(-)$ provided the strong form of the Künneth theorem is satisfied for any finite smash product of the spaces given by

- (1) X_i ,
- (2) A_i , and
- (3) $|K_I|$ for all $I \subseteq [m]$.

The analogue of Theorem 1.9 for any cohomology theory follows next.

Theorem 1.12. *Let $(\underline{CX}, \underline{X}) = \{(CX_i, X_i)\}_{i=1}^m$ be a family of based CW-pairs, and K be an abstract simplicial complex with m vertices which is proper with respect to a cohomology theory $E^*(-)$. Then the cup-product structure for the cohomology algebra $E^*(Z(K; (\underline{CX}, \underline{X})))$ is a functor of*

- (1) the cohomology algebras $E^*(X_i)$, $1 \leq i \leq m$, and
- (2) the cohomology algebras $E^*(Z(K_I; (D^1, S^0)))$ for all $I \subseteq K$.

Furthermore, there are natural isomorphisms of multiplicatively closed sub-modules of

$$E^*(\Sigma|K_I|) \otimes E^*(X_1) \otimes \cdots \otimes E^*(X_m)$$

given by

$$\tilde{E}^*(\widehat{Z}(K_I; (\underline{CX}, \underline{X})_I)) \rightarrow \tilde{E}^*(\Sigma|K_I|) \otimes \tilde{E}^*(\widehat{X}^I),$$

and

$$\tilde{E}^*(\Sigma|K_I|) \otimes \tilde{E}^*(\widehat{X}^I) \rightarrow \tilde{E}^*(\Sigma|K_I|) \otimes \tilde{E}^*(X_{i_1}) \otimes \tilde{E}^*(X_{i_2}) \otimes \cdots \otimes \tilde{E}^*(X_{i_k})$$

for $I = (i_1, \dots, i_k) \subseteq [m]$.

In addition, there are natural isomorphisms

$$\bigoplus_{I \subseteq [m]} \tilde{E}^*(\widehat{Z}(K_I; (\underline{CX}, \underline{X})_I)) \rightarrow \tilde{E}^*(Z(K; (\underline{CX}, \underline{X})))$$

where $\tilde{E}^*(\widehat{Z}(K_I; (\underline{CX}, \underline{X})_I))$ is isomorphic to a multiplicatively closed sub-module of

$$\tilde{E}^*(Z(K; (\underline{CX}, \underline{X}))).$$

The definition of *partial diagonals* is the subject of section 2. Information about partial diagonals is extended to smash product generalized moment-angle complexes in section 3. Sections 4, 5, and 6 give the proofs of Theorems 1.4, 1.6, and 1.8 respectively.

Partial diagonals are given in section 7 for the smash moment-angle complexes $\widehat{Z}(K; (\underline{CX}, \underline{X}))$ where CX is the cone on the CW-complex X . In this case, the cohomology algebra of

$Z(K; (CX, X))$ is shown to be a functor, with mild restrictions, of the cohomology algebra for X and for $Z(K; (D^1, S^0))$. This result is a counter-point to Theorem 2.35 of [1] where an analogous result is proven for $Z(K; (X, *))$.

CONTENTS

| | |
|---|----|
| 1. Introduction, definitions, and main results | 1 |
| 2. The partial diagonal in product spaces | 8 |
| 3. The partial diagonal in the smash moment-angle complexes | 10 |
| 4. Proof of Theorem 1.4 | 11 |
| 5. Proof of Theorem 1.6 | 11 |
| 6. Proof of Theorem 1.8 | 12 |
| 7. The partial diagonal for (CX, X) | 12 |
| 8. Acknowledgements | 14 |
| References | 14 |

2. The partial diagonal in product spaces

Let $Y^{[m]} = Y_1 \times \cdots \times Y_m$ and $\widehat{Y}^I = Y_{i_j} \wedge \cdots \wedge Y_{i_k}$ for $I = (i_1, \dots, i_k) \subseteq [m]$. There are natural projection maps $\widehat{\Pi}_I : Y^{[m]} \rightarrow \widehat{Y}^I$ obtained as the composition

$$Y^{[m]} \xrightarrow{\Pi_I} Y^I \xrightarrow{\rho_I} \widehat{Y}^I$$

where Π_I is the projection map and ρ_I is the quotient map.

Let

$$\widehat{\Delta}_I : \widehat{Y}^I \rightarrow \widehat{Y}^I \wedge \widehat{Y}^I$$

be the reduced diagonal map of \widehat{Y}^I , and define

$$C_I = \{(J, L) \mid J, L \subseteq I \text{ and } J \cup L = I\}.$$

Construct

$$(2.7) \quad \widehat{\Delta}_I^{J,L} : \widehat{Y}^I \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L$$

as follows. Let

$$W_I^{J,L}$$

denote the smash product

$$\bigwedge_{\ell(J)+\ell(L)} W_i,$$

where

$$W_i = \begin{cases} Y_i & \text{if } i \in I - (J \cap L) \\ Y_i \wedge Y_i & \text{if } i \in J \cap L. \end{cases}$$

Note that if $J \cap L = \emptyset$, then $W_I^{J,L} = \widehat{Y}^I$.

Define

$$\psi : \widehat{Y}^I \rightarrow W_I^{J,L} \quad \text{as} \quad \psi = \bigwedge_{i \in I} \psi_i$$

and $\psi_j : Y_j \rightarrow W_j$ as

$$\psi_j = \begin{cases} \text{Id} & \text{if } i \in I - (J \cap L) \\ \Delta_i : Y_i \rightarrow Y_i \wedge Y_i & \text{if } i \in J \cap L \end{cases}$$

where Δ_i is the reduced diagonal of Y_i .

Observe that the smash products $W_I^{J,L}$, and $\widehat{Y}^J \wedge \widehat{Y}^L$ have the same factors, but in a different order arising from the natural shuffles. So let

$$s : \widehat{Y}^J \wedge \widehat{Y}^L \rightarrow W_I^{J,L}$$

denote the natural homeomorphism given by a shuffle. Let

$$\theta : W_I^{J,L} \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L$$

denote the inverse of s . Then define

$$\widehat{\Delta}_I^{J,L} : \widehat{Y}^I \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L$$

as the composition

$$(2.8) \quad \widehat{Y}^I \xrightarrow{\psi} W_I^{J,L} \xrightarrow{\theta} \widehat{Y}^J \wedge \widehat{Y}^L.$$

Further, observe that there is a commutative diagram

$$(2.9) \quad \begin{array}{ccc} Y^{[m]} & \xrightarrow{\Delta_{[m]}} & Y^{[m]} \wedge Y^{[m]} \\ \downarrow \widehat{\Pi}_J & & \downarrow \widehat{\Pi}_J \wedge \widehat{\Pi}_L \\ \widehat{Y}^I & \xrightarrow{\widehat{\Delta}_I^{J,L}} & \widehat{Y}^J \wedge \widehat{Y}^L. \end{array}$$

Next specialize Definition 1.3 to the case of $(\underline{X}, \underline{A}) = (\underline{X}, \underline{X}) = \{(X_i, X_i)\}_{i=1}^m$ with $K = \emptyset$. In this case, let

$$\mathcal{H}^*(K; (\underline{X}, \underline{A})) = \mathcal{H}^*(X^{[m]}).$$

Then the $*$ -product in $\mathcal{H}^*(X^{[m]})$ is given by the composition

$$H^p(X^J) \otimes H^q(X^L) \xrightarrow{\rho^*} H^{p+q}(X^J \wedge X^L) \xrightarrow{(\widehat{\Delta}_I^{J,L})^*} H^{p+q}(\widehat{X}^I)$$

where $I = J \cup L$. The notation $u * v$ denotes the $*$ -product of classes $u \in H^p(\widehat{X}^J), v \in H^q(\widehat{X}^L)$; the class $u * v$ is a product of these two cohomology classes. Furthermore,

$$\mathcal{H}^n(Y^{[m]}) \cong \bigoplus_I H^n(\widehat{Y}^I)$$

as given in Definition 1.3.

Diagram (2.9) implies that

$$(2.10) \quad \Pi_I^*(u * v) = (\Pi_J^* u) \smile (\Pi_L^* v)$$

where \smile is the cup-product in $H^*(X^{[m]})$. The well-known splitting of the suspension of a product stated in [1] gives an additive isomorphism

$$\eta : \mathcal{H}^*(X^{[m]}) \rightarrow H^*(X^{[m]}).$$

Since $\eta|_{H^*(\widehat{Y}\widehat{X}^I)} = (\Pi_I)^*$, diagram (2.9) implies that η is a ring isomorphism.

The next result, a special case of Theorem 1.4, follows.

Theorem 2.1. *Let $(\underline{X}, \underline{X}) = \{(X_i, X_i)\}_{i=1}^m$ with $K = \emptyset$. Then, the mapping*

$$\eta : \mathcal{H}^*(X^{[m]}) \rightarrow H^*(X^{[m]})$$

is a ring isomorphism.

This special case of Theorem 1.4 will be extended to pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ in the next section.

3. The partial diagonal in the smash moment-angle complexes

In this section, the partial diagonal $\widehat{\Delta}_I^{J,L} : \widehat{Y}^I \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L$ of section 2 is extended to smash moment-angle complexes as follows. Let K be a simplicial complex with m vertices, $\sigma \in K$ simplices of K , and $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}$. The simplices of K_I are $\sigma \cap I$, and $(\underline{X}, \underline{A})_I$ the associated family. Using the notation of section 1,

$$\widehat{Z}(K_I; (\underline{X}, \underline{A})_I) = \bigcup_{\sigma \in K} \widehat{Y}^I(\sigma \cap I) \subset \bigcup_{\sigma \in K} \widehat{X}^I(\sigma \cap I).$$

Similarly

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \bigcup_{\sigma \in K} Y(\sigma) \subset \bigcup_{\sigma \in K} X^I(\sigma).$$

Observe that to give maps out of

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \operatorname{colim}(D(\sigma)),$$

it suffices to give compatible maps out of each space $D(\sigma)$. In addition, to give maps out of

$$\widehat{Z}(K_I; (\underline{X}, \underline{A})_I) = \bigcup_{\sigma \in K} \widehat{Y}^I(\sigma \cap I) = \operatorname{colim}(\widehat{Y}^I(\sigma \cap I)),$$

it suffices to give compatible maps out of each space $\widehat{Y}^I(\sigma \cap I)$. Observe that the maps

$$\widehat{\Delta}_I^{J,L} : \widehat{X}^I \rightarrow \widehat{X}^J \wedge \widehat{X}^L$$

as given in the composition (2.8) restrict to maps

$$(3.11) \quad \widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \rightarrow \widehat{Z}(K_J) \wedge \widehat{Z}(K_L).$$

4. Proof of Theorem 1.4

The proof of Theorem 1.2 in [1] gives that suspending and adding the maps $\widehat{\Pi}_I$ provides a map

$$\Sigma(\vee \widehat{\Pi}_I) : \Sigma Z(K) \rightarrow \Sigma \vee_{I \subseteq [m]} \widehat{Z}(K_I)$$

which is a homotopy equivalence. Furthermore, each map

$$\widehat{\Pi}_I : Z(K) \rightarrow \widehat{Z}(K_I)$$

induces a morphism of cohomology algebras while the sum of these maps induces

$$\eta : \mathcal{H}^*(K; (\underline{X}, \underline{A})) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})))$$

which is an additive isomorphism. Since η restricted to $H^*(\widehat{K}_I)$ is $\widehat{\Pi}_I^*$, this implies that η is an algebra isomorphism, the statement of Theorem 1.4.

Note: It is unnecessary to prove the associativity and the graded commutativity of the $*$ -product directly. Those properties are a consequence of the isomorphism in Theorem 1.4.

5. Proof of Theorem 1.6

If Y_i is a suspension space then the reduced diagonal $\Delta_i : Y_i \rightarrow Y_i \wedge Y_i$ is null-homotopic. Thus if $J \cap L \neq \emptyset$, the map $\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \rightarrow \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$ is null-homotopic. Theorem 1.6 follows.

6. Proof of Theorem 1.8

The first part of the Theorem is a consequence of the fact that there is a homotopy equivalence

$$\widehat{Z}(K_I; (\underline{\Sigma^T X}, \underline{\Sigma^T A})_I) \rightarrow \Sigma^{d(T_I)} \widehat{Z}(K_I; (\underline{X}, \underline{A})_I)$$

where for each $T = (t_1, \dots, t_m)$,

$$d(T_I) = \Sigma_{u \in I} t_i,$$

so

$$\widehat{Z}(K_I; (\underline{\Sigma^T X}, \underline{\Sigma^T A})_I)$$

and

$$\widehat{Z}(K_I; (\underline{\Sigma^{T'} X}, \underline{\Sigma^{T'} A}))$$

are stably equivalent.

The second part follows from the fact that all possible signs that may come from permutations will have the same signs as $T \equiv T' \pmod{2}$.

7. The partial diagonal for (CX, X)

The purpose of this section is to prove Theorem 1.9 which exhibits the algebra structure for the cohomology of $Z(K; (\underline{CX}, \underline{X}))$. Let K be an abstract simplicial complex with m vertices. Given families of based CW-pairs $(\underline{Y}, \underline{B}) = \{(Y_i, B_i)\}_{i=1}^m$, and $(\underline{W}, \underline{C}) = \{(W_i, C_i)\}_{i=1}^m$, consider the natural shuffle map

$$(7.12) \quad \text{shuff} : (Y_1 \times W_1) \times \cdots \times (Y_m \times W_m) \rightarrow (Y_1 \times \cdots \times Y_m) \times (W_1 \times \cdots \times W_m)$$

restricted to

$$(7.13) \quad \text{shuff} : Z(K; (\underline{Y} \times \underline{W}, \underline{B} \times \underline{C})) \rightarrow Z(K; (\underline{Y}, \underline{B})) \times Z(K; (\underline{W}, \underline{C})).$$

Let $(Y, B) = (D^1, S^0)$, and $(\underline{W}, \underline{C}) = (\underline{X}, \underline{X})$. There is an induced map

$$(7.14) \quad \gamma : \widehat{Z}(K; (\underline{D^1} \wedge \underline{X}, \underline{S^0} \wedge \underline{X})) \rightarrow \widehat{Z}(K; (\underline{D^1}, \underline{S^0})) \wedge \widehat{Z}(K; (\underline{X}, \underline{X}))$$

where $Z(K; (\underline{X}, \underline{X})) = X_1 \wedge \cdots \wedge X_m = \widehat{X}^{[m]}$. Furthermore,

$$\widehat{Z}(K; (\underline{D^1} \wedge \underline{X}, \underline{S^0} \wedge \underline{X})) = \widehat{Z}(K; (\underline{CX}, \underline{X})).$$

The next Lemma follows by inspection.

Lemma 7.1. *Let K be an abstract simplicial complex with m vertices. The natural map induced by the shuffle*

$$\gamma : \widehat{Z}(K; (\underline{D}^1 \wedge \underline{X}, \underline{S}^0 \wedge \underline{X})) \rightarrow Z(K; (\underline{D}^1, \underline{S}^0)) \wedge Z(K; (\underline{X}, \underline{X}))$$

is a homeomorphism. Thus $\widehat{Z}(K; (\underline{D}^1 \wedge \underline{X}, \underline{S}^0 \wedge \underline{X})) = \widehat{Z}(K; (\underline{CX}, \underline{X}))$ is naturally homeomorphic to

$$\widehat{Z}(K; (\underline{D}^1, \underline{S}^0)) \wedge \widehat{X}^{[m]}.$$

Let $\widehat{Z}(K_I)$ denote $\widehat{Z}(K_I; (\underline{CX}, \underline{X}))$ and let

$$\gamma_I : \widehat{Z}(K_I; (\underline{CX}, \underline{X})) \rightarrow \widehat{Z}(K_I; (\underline{D}^1, \underline{S}^0)) \wedge \widehat{X}^{K_I}$$

denote the homeomorphism of Lemma 7.1. Observe that if $J \cup L = I$, there is a homotopy commutative diagram

$$\begin{array}{ccc} \widehat{Z}(K_I) & \xrightarrow{\widehat{\Delta}_I^{J,L}} & \widehat{Z}(K_J) \wedge \widehat{Z}(K_L) \\ \gamma_I \downarrow & & \downarrow \gamma_J \wedge \gamma_L \\ \widehat{Z}(K_I; (D^1, S^0)) \wedge \widehat{X}^I & \xrightarrow{\widehat{\Psi}_I^{J,L}(\underline{CX}, \underline{X})} & \widehat{Z}(K_J; (D^1, S^0)) \wedge \widehat{X}^J \wedge \widehat{Z}(K_L; (D^1, S^0)) \wedge \widehat{X}^L \end{array}$$

where the map $\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \rightarrow \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$ is given by (3.11).

Specialize to $(CX, X) = (D^1, S^0)$ with $\widehat{Z}(K_I) = \widehat{Z}(K_I; (\underline{CX}, \underline{X}))$ as well as to the unique pointed homeomorphism $X \wedge \cdots \wedge X \rightarrow X = S^0$ to obtain a homotopy commutative diagram

$$\begin{array}{ccc} \widehat{Z}(K_I) & \xrightarrow{\widehat{\Delta}_I^{J,L}} & \widehat{Z}(K_J) \wedge \widehat{Z}(K_L) \\ \alpha_I \downarrow & & \downarrow \alpha_J \wedge \alpha_L \\ \widehat{Z}(K_I; (D^1, S^0)) & \xrightarrow{\widehat{\Psi}_I^{J,L}(D^1, S^0)} & \widehat{Z}(K_J; (D^1, S^0)) \wedge \widehat{Z}(K_L; (D^1, S^0)). \end{array}$$

This diagram is used to address the general case.

Consider $(\underline{CX}, \underline{X})$ together with the map

$$\widehat{Z}(K_I; (D^1, S^0)) \wedge \widehat{X}^I \xrightarrow{\widehat{\Psi}_I^{J,L}(\underline{CX}, \underline{X})} \widehat{Z}(K_J; (D^1, S^0)) \wedge \widehat{X}^J \wedge \widehat{Z}(K_L; (D^1, S^0)) \wedge \widehat{X}^L.$$

Observe that the map $\widehat{\Psi}_I^{J,L}(\underline{CX}, \underline{X})$ is given by the composite

$$\widehat{Z}(K_I; (D^1, S^0)) \wedge \widehat{X}^I \xrightarrow{\widehat{\Psi}_I^{J,L}(D^1, S^0) \wedge \widehat{\Delta}_I^{J,L}} \widehat{Z}(K_J; (D^1, S^0)) \wedge \widehat{X}^J \wedge \widehat{Z}(K_L; (D^1, S^0)) \wedge \widehat{X}^L$$

with

$$\widehat{Z}(K_J; (D^1, S^0)) \wedge \widehat{X}^J \wedge \widehat{Z}(K_L; (D^1, S^0)) \wedge \widehat{X}^L \xrightarrow{1 \wedge \tau \wedge 1} \Sigma|K_J| \wedge \widehat{X}^J \wedge \Sigma|K_L| \wedge \widehat{X}^L$$

where

$$\tau : \widehat{Z}(K_L; (D^1, S^0)) \wedge \widehat{X}^J \rightarrow \widehat{X}^J \wedge \widehat{Z}(K_L; (D^1, S^0))$$

is the natural map which swaps factors. This last assertion follows by inspection of the definitions. This suffices to prove Theorems 1.9, and 1.12.

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